Perturbation theory of boson dynamical systems

1990 J. Phys. A: Math. Gen. 23 L783

(http://iopscience.iop.org/0305-4470/23/16/001)

J. Phys. A: Math. Gen. 23 (1990) L783–L788. Printed in the UK

LETTER TO THE EDITOR

Perturbation theory of boson dynamical systems

Anton Amann

Laboratory of Physical Chemistry, Eidgenössische Technische Hochshule-Zentrum, CH-8092 Zürich, Switzerland

Received 4 June 1990

Abstract. Dynamical evolutions of boson (spin-boson and related) systems based on the widely used canonical commutation relations C^* algebra have a shortcoming. The dynamical group is not pointwise norm-continuous and therefore cannot be perturbed by a (bounded self-adjoint) element of the underlying abstract C^* algebra. Hence it is necessary to define notions like 'invariant state', 'ground state', 'KMS state', etc with respect to the perturbed dynamics in suitable Hilbert space representations. here the original dynamical system is supplemented by an auxiliary pointwise norm-continuous dynamical system in such a way that invariant states of the original system correspond to invariant states of the auxiliary system. This bijective correspondence is sequentially continuous and preserves the KMS (ground state) characterizing conditions. As a typical application it is verified that Spohn's ground states of the spin-boson model (arising as a temperature to zero limit of thermic equilibrium states) are ground states in the algebraic sense, i.e. are eigenstates of the respective (Gel'fand–Naimark–Segal) Hamiltonian with an eigenvalue at the lower end of the Hamiltonian spectrum.

A boson system (cf [1], section 5.2.2.2) is characterized by a pre-Hilbert space $\mathscr{K}$ with scalar product $\langle\cdot|\cdot\rangle$. It's C^* algebra $\Delta(\mathscr{K})$ is generated by unitary 'Weyl' operators $W(f), f\in\mathscr{K}$, with the canonical commutation relations

$$W(f)\,W(f') = W(f+f')\exp\{-\mathrm{i}\,\mathrm{Im}\langle f|f'\rangle\} \qquad f,f'\in\mathscr{K} \tag{1}$$

where $\mathrm{Im}\langle f|f'\rangle$ is the imaginary part of the scalar product $\langle f|f'\rangle$. The C^* algebra $\Delta(\mathscr{K})$ is a concise reformulation of Heisenberg's commutation relations (or boson commutation relations), particularly convenient for systems with infinitely many degrees of freedom.

A C^* algebra $\mathscr{A}$ is characterized by a norm $\|\cdot\|$ ('length' of operators) fulfilling

$$\|A^*A\| = \|A\|^2 \qquad A\in\mathscr{A}. \tag{2}$$

This norm is mathematically useful but the respective norm topology has no particular physical significance. The physically relevant topology is the σ-weak topology mentioned below, which is defined with respect to expectation values.

Representations of a C^* algebra on a Hilbert space bring in a particular physical context. Physically inequivalent representations arise if the C^* algebra describes a system with infinitely many degrees of freedom. For a boson C^* algebra $\Delta(\mathscr{K})$ the Fock representation plays a role if one studies ground states with respect to a quasi-free time evolution. Equilibrium states (KMS states with non-zero temperature) of a boson system already use a representation inequivalent to the Fock representation. Neither ground states nor equilibrium states of spin-boson models [2–4] can generally be

0305-4470/90/160783+06$03.50

described in a Fock space representation. The use of phsyically inequivalent representations of infinite sytems broadens the scope of quantum mechanical theory. It allows the description and even the derivation of superselection rules (cf [2, 5–8]).

The physically relevant observables in a specific representation $(\mathscr{H}, \pi)$ arise as limits of (represented) C^*-algebra operators. An operator T acting on the Hilbert space $\mathscr{H}$ is said to be the limit of a sequence $(\pi(A_j))_{j\in\mathbb{N}}$, where the operators A_j are elements of the underlying C^* algebra $\mathscr{A}$, if convergence

$$\lim_{j\to\infty} \operatorname{Tr}(\pi(A_j)D) = \operatorname{Tr}(TD) \tag{3}$$

holds with respect to an arbitrary positive trace-class operator D acting on the representation Hilbert space (σ-weak limit). Here ⟪Tr(·)⟫ denotes the trace of an operator. Limits in the σ-weak sense are defined with respect to expectation values and are therefore physically relevant. The set of all σ-weak limits of C^*-algebra operators in a certain representation form a von Neumann algebra (W^* algebra) denoted by $\{\pi(\mathscr{A})\}''$. As examples for the limit process (3) consider the construction of a global magnetization observable or a global momentum observable starting with local ones.

The C^* algebra of a 'small' system coupled to a boson field is given as the tensor product of the respective C^* algebras. Consider as an example a spin-boson system consisting of *one* spin-$\frac{1}{2}$ and infinitely many bosons (cf [3, 4, 9–14]): the relevant C^* algebra is a tensor product $\mathscr{M}_2 \otimes \Delta(\mathscr{H})$ of the algebra $\mathscr{M}_2$ of the 2×2 matrices and a suitable boson algebra $\Delta(\mathscr{H})$.

The Hamiltonian of this spin-boson system consists of three parts, referring to the spin, the field and the coupling between them. The field part together with the coupling corresponds to a dynamical automorphic group α^0 of the C^* algebra $\mathscr{M}_2 \otimes \Delta(\mathscr{H})$ [4]. The Hamiltonian $\varepsilon\sigma_1$ of the isolated spin is considered as a perturbation (σ_1 is a Pauli matrix and 2ε is the level splitting).

Heisenberg dynamical (automorphic) evolutions

$$t \to \alpha_t(A) \qquad t \in \mathbb{R}, A \in \mathscr{A} \tag{4}$$

of a boson-type C^* algebra $\mathscr{A}$ are frequently built up from a quasifree evolution sending Weyl operators into Weyl operators. Consider the dynamics α_0 as an example. Since arbitrary distinguished Weyl operators $W(f_1)$ and $W(f_2)$ fulfil ([1], theorem 5.2.8.):

$$\| W(f_1) - W(f_2) \| = 2 \tag{5}$$

one cannot expect the mappings (4) to be norm-continuous. In mathematics and mathematical physics, on the other hand, this sort of norm-continuity is often assumed to hold for dynamical groups of C^* algebras. The respective C^* system (consisting of algebra and dynamics) is then called pointwise norm-continuous.

The perturbation of a dynamical group by a bounded self-adjoint operator is no problem for pointwise norm-continuous dynamical groups (cf [1], section 5.4). For C^* systems without this property, such as the spin-boson system with the dynamics α^0 of above, one *cannot* properly define a perturbed dynamics of the underlying C^* algebra. That this is not just due to technical problems can be inferred from the examples given in [15].

As a consequence, even invariant states with respect to the perturbed dynamics cannot be defined abstractly (=Hilbert space free) but only with respect to a suitable representation, e.g., its associated Gel'fand–Naimark–Segal representation (in short

GNS representation, see [16], sections 2.3.16 and 2.3.17). The same remark holds *a forteriori* for KMS states, ground states, etc.

Definition 1. Consider a C^* algebra $\mathscr{A}$ and a dynamical (automorphic) group $\{\alpha_t | t \in \mathbb{R}\}$ which is not (necessarily) pointwise norm-continuous. Let P be a self-adjoint element of $\mathscr{A}$. Then a state ϕ on $\mathscr{A}$ is called invariant under the perturbed dynamics 'α^P' if there exists a representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, a self-adjoint operator H acting on $\mathscr{H}$ and an eigenvector ξ of H in $\mathscr{H}$ such that

$$\pi(\alpha_t(A)) = \exp\{\mathrm{i}(H-P)t\}\pi(A)\exp\{-\mathrm{i}(H-P)t\} \qquad t \in \mathbb{R} \tag{6}$$

$$\phi(A) = \langle \xi | \pi(A)\xi \rangle \qquad A \in \mathscr{A}. \tag{7}$$

The 'Hamiltonian' $(H-P)$ corresponds to the dynamics α, whereas the perturbed Hamiltonian $H=(H-P)+P$ should implement the perturbed dynamics 'α^P'. The latter is written in quotation marks, since one has no guarantee that it defines any dynamics on $\mathscr{A}$, that is

$$\exp\{\mathrm{i}Ht\}\pi(\mathscr{A})\exp\{-\mathrm{i}Ht\} \not\subseteq \pi(\mathscr{A}) \tag{8}$$

may arise [15]. Nevertheless, the associated von Neumann algebra $\{\pi(\mathscr{A})\}''$ fulfils

$$\exp\{\mathrm{i}Ht\}\{\pi(\mathscr{A})\}''\exp\{-\mathrm{i}Ht\} = \{\pi(\mathscr{A})\}''. \tag{9}$$

Therefore it is possible to define a perturbed dynamics $\tilde{\alpha}^P$ on the von Neumann algebra $\{\pi(\mathscr{A})\}''$ by

$$\tilde{\alpha}^P_t(X) := \exp\{\mathrm{i}Ht\}X\exp\{-\mathrm{i}Ht\} \qquad X \in \{\pi(\mathscr{A})\}'', t \in \mathbb{R}. \tag{10}$$

As an example for a perturbation P consider the Hamiltonian $\varepsilon\sigma_1$ of the (isolated) spin in the spin-boson model above.

The property of being an 'α^P'-invariant state can be checked in the GNS representation; that is, the representation π of definition 1 can be replaced by the GNS representation π_ϕ. Hence special classes of invariant states (KMS state, ground states, ...) can be defined with respect to the GNS-von Neumann algebra $\{\pi_\phi(\mathscr{A})\}''$ and the respective dynamics (9). For ground states, e.g., definition 1 has to be modified only slightly by saying that the eigenvector ξ has an eigenvalue at the lower end of the Hamiltonian's spectrum. Such ground states are called 'ground states in the algebraic sense'.

The problem with these definitions is that varous 'obvious' interrelations cannot be proven. Consider, e.g., a sequence $(\omega_n)_{n\in\mathbb{N}}$ of β_n-KMS states with respect to the perturbed dynamics 'α^P'. Assume that this sequence converges to a state ϕ_0 with respect to expectation values

$$\lim_{n\to\infty} \omega_n(A) = \phi_0(A) \qquad A \in \mathscr{A}. \tag{11}$$

One would expect ϕ_0 to be a ground state in the sense defined in the last paragraph. To prove this for pointwise norm-continuous C^* systems does not present any problems. Perturbation theory is then defined abstractly (without reference to representations) and the powerful tool of a generator δ of the dynamics α allows us to give this proof fairly easily (cf [1], proposition 5.3.23).

Ground states of the spin-boson model are introduced by [3] in this way. Apart from the Hamiltonian of the isolated spin an additional perturbation $h\sigma_3$ is used, where σ_3 is a Pauli matrix and h is the strength of the pertrubation. It is finally set to zero

($h \to \pm 0$) and thereby offers the chance to get two different ground states and hence break the symmetry of the spin-boson model. Again, pointwise norm-continuity *would* allow to prove that these limits give rise to ground states in the algebraic sense.

The propositions below show that these difficulties can be overcome. A given C^* system without norm-continuity properties can be replaced—for certain purposes—by an auxiliary pointwise norm-continuous C^* system. The auxiliary system admits an infinitesimal generator and hence allows to transfer various proofs already known for pointwise norm-continuous C^* systems. The continuity properties necessary are introduced in the following definition.

Definition 2. Consider a C^* algebra $\mathscr{A}$ and a dynamical (automorphic) group $\{\alpha_t | t \in \mathbb{R}\}$ which is not (necessarily) pointwise norm-continuous. Then a state ϕ on $\mathscr{A}$ is called α-continuous if the mappings

$$t \to \phi(A\alpha_t(B)C) \qquad t \in \mathbb{R} \tag{12}$$

are continuous for arbitrary elements A, B and C in $\mathscr{A}$.

This sort of continuity condition refers to a physically relevant topology and is rather weak. It should be fulfilled by any physically relevant state. Any invariant state in the sense of definition 1 is α-continuous. Furthermore, any state of a pointwise norm-continuous C^* system is automatically α-continuous.

The propositions below generalize the results of [17] on invariant states of C^* systems without pointwise norm-continuity properties. Proofs will be given elsewhere. The main difficulty is settled in the following proposition.

Proposition. Let $(\mathscr{A}, \mathbb{R}, \alpha)$ be a C^* system which is not (necessarily) pointwise norm-continuous. Consider a self-adjoint perturbation $P \in \mathscr{A}$ of the dynamics α. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of 'α^P'-invariant states converging to an α-continuous state ϕ_0. Then it follows that ϕ_0 is again 'α^P' invariant.

The result of this proposition is completely trivial for a pointwise norm-continuous C^* system. It paves the way for the following main theorem and its corollaries.

Theorem. Let $(\mathscr{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system where the action α is not (necessarily) pointwise norm-continuous and consider a perturbation of this action by a self-adjoint operator P from $\mathscr{A}$. Then there exist:

(i) *a* pointwise norm-continuous C^* system $(\mathscr{D}^P, \mathbb{R}, \theta^P)$;

(ii) a representation μ of $\mathscr{D}^P$ on a Hilbert space $\mathscr{H}_\mu$;

(iii) and an affine sequentially continuous bijection between the space of 'α^P'-invariant states on $\mathscr{A}$ and the space of θ^P-invariant states $\tilde{\phi}$ on $\mathscr{D}^P$ which can be implemented by a trace-class operator D acting on $\mathscr{H}_\mu$.

Consider invariant states ϕ and $\tilde{\phi}$ on $\mathscr{A}$ and $\mathscr{D}^P$, corresponding under the above bijection, and denote by $(\mathscr{H}_\phi, \pi_\phi, \xi_\phi)$ and $(\mathscr{H}_{\tilde{\phi}}, \pi_{\tilde{\phi}}, \xi_{\tilde{\phi}})$ the associated GNS representations. Then one can identify (modulo unitary equivalence) $\mathscr{H}_\phi$ and $\mathscr{H}_{\tilde{\phi}}$, ξ_ϕ and $\xi_{\tilde{\phi}}$, the von Neumann algebras $\{\pi_{\tilde{\phi}}(\mathscr{D})\}''$ and $\{\pi_\phi(\mathscr{A})\}''$ and the respective dynamics on these von Neumann algebras. Hence ϕ is a β-KMS state (ground state) with respect to the action 'α^P' if and only if $\tilde{\phi}$ is a β-KMS state (ground state) with respect to the action θ^P.

Remark. (*a*) A state $\tilde{\phi}$ on $\mathscr{D}^P$ is said to be implemented by a trace-class operator D acting on $\mathscr{H}_\mu$ if

$$\tilde{\phi}(B) = \mathrm{Tr}(D\mu\{B\}) \qquad B \in \mathscr{D}^P \tag{13}$$

holds.

(*b*) The dynamics on the von Neumann algebra $\{\pi_\phi(\mathscr{A})\}''$ is defined via (10). For the respective dynamics on $\{\pi_{\tilde{\phi}}(\mathscr{D})\}''$ the corresponding construction coincides with the ordinary GNS dynamics ([16], section 2.3.17).

Corollary 1. Let $(\mathscr{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system where the action α is not (necessarily) pointwise norm-continuous. Consider a sequence $(\omega_n)_{n\in\mathbb{N}}$ of β_n-KMS states with respect to the perturbed (pseudo) dynamics 'α^P', converging to a state ϕ_0 in the weak *-topology. Assume that the state ϕ_0 is α-continuous, and that

$$\lim_{n\to\infty} \beta_n = \beta \tag{14}$$

exists in $\mathbb{R} \cup \{\infty\}$. It follows that ϕ_0 is a β-KMS state (ground state, if $\beta = \infty$).

Proof: Use ([1], proposition 5.3.23).

Corollary 2. Let $(\mathscr{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system where the action α is not (necessarily) pointwise norm-continuous. Consider a sequence $(\phi_n)_{n\in\mathbb{N}}$ of ground states with respect to a perturbed (pseudo) dynamics 'α^P', converging to a state ϕ_0 in the weak *-topology. Assume that the state ϕ_0 is α-continuous. It follows that ϕ_0 is a ground state.

By explicit computation, one may show that the ground states of the spin-boson model as defined by Spohn in [3] are α^0 continuous (see definition 2). Here α^0 is the dynamics of the spin-boson model without the Hamiltonian part of the isolated spin. The latter is considered as a perturbation. Some minor modifications of the theorem (incorporating the additional perturbation $h\sigma_3$ mentioned above) then lead to:

Corollary 3. Spohn's ground states of the spin-boson model are ground states in the algebraic sense, i.e. are eigenstates of the respective (Gel'fand-Naimark-Segal) Hamiltonian with an eigenvalue at the lower end of the Hamiltonian's spectrum.

References

[1] Bratteli O and Robinson D W 1981 *Operator Algebras and Quantum Statistical Mechanics* vol 2 (Springer: New York)

[2] Pfeifer P 1980 Chiral Molecules—a Superselection Rule Induced by the Radiation Field *Thesis* ETH-Zürich No 6551, ok Gotthard S+D AG

[3] Spohn H 1989 *Ground state(s) of the spin-boson Hamiltonian Commun. Math. Phys.* **123** 277-304

[4] Fannes M, Nachtergaele B and Verbeure A 1988 The equilbirium states of the spin-boson model *Commun. Math. Phys.* **114** 537-48

[5] Primas H 1983 *Chemistry, Quantum Mechnics, and Reductionism. Perspectives in Theoretical Chemistry* (Berlin: Springer) 2nd edn

[6] Strocchi F 1985 *Elements of Quantum Mechanics of Infinite Systems* (Singapore: World Scientific)

[7] Amann A and Müller-Herold U 1986 Momentum operators for large systems *Helv. Phys. Acta* **59** 1311-20

[8] Amann A 1988 Chirality as a classical observable in algebraic quantum mechanics *Fractals, Quasicrystals, Chaos, Knots and Algebraic Quantum Mechanics* ed A Amann, L Cederbaum and W Gans (Dordrecht: Kluwer) pp 305-25

[9] Spohn H and Dümcke R 1985 Quantum tunneling with dissipation and the Ising model over R *J. Stat. Phys.* **41** 389-423
[10] Nachtergaele B 1987 Exakte resultaten voor het Spin-Boson model *Thesis*, Katholieke Universiteit Leuven
[11] Fannes M, Nachtergaele B and Verbeure A 1987 Quantum tunneling in the spin-boson model *Europhys. Lett.* **4** 963-5
[12] Fannes M, Nachtergaele B and Verbeure A 1988 Tunneling in the equilibrium state of a spin-boson model *J. Phys. A: Math. Gen.* **21** 1759-68
[13] Fannes M and Nachtergaele B 1988 Translating the spin-boson model into a classical system *J. Math. Phys.* **29** 2288-93
[14] Fannes M 1989 Temperature states of spin-boson models *Quantum Probability and Applications IV (Lecture Notes in Mathematics 1396)* ed L Accardi and W von Waldenfels (Berlin: Springer)
[15] Fannes M and Verbeure A 1974 On the time evolution automorphisms of the CCR-algebra for quantum mechanics *Commun. Math. Phys.* **35** 257-64
[16] Bratteli O and Robinson D W 1987 *Operator Algebras and Quantum Statistical Mechanics* vol 1 (New York: Springer) 2nd edn
[17] Amann A 1990 Invariant states of C^*-systems without norm-continuity properties *Preprint*